Year 12 Mathematics

IAS 2.13

Simulations

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Contents

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NCEA 2 Internal Achievement Standard 2.13 – Simulations

This achievement standard involves investigating a situation involving elements of chance using a simulation.

Achievement	Achievement with Merit	Achievement with Excellence
• Investigate a situation involving elements of chance using a simulation.	• Investigate a situation involving elements of chance using a simulation, with justification.	• Investigate a situation involving elements of chance using a simulation, with statistical insight.

- ◆ This achievement standard is derived from Level 7 of The New Zealand Curriculum, Learning Media and is related to the achievement objective
  - ❖ Investigate situations that involve elements of chance
    - • calculating probabilities using such tools as simulations and technology;

  in the Statistics strand of the Mathematics and Statistics Learning Area.

- ◆ Investigate a situation involving elements of chance using a simulation involves showing evidence of using each component of the simulation process.

  Investigate a situation involving elements of chance using a simulation, with justification involves linking components of the simulation process to the context, explaining relevant considerations made in the design of the simulation and supporting findings with statements which refer to evidence gained from the simulation.

  Investigate a situation involving elements of chance using a simulation, with statistical insight involves integrating statistical and contextual knowledge throughout the simulation process, which may involve reflecting about the process or considering other variables.

- ◆ The process of carrying out a simulation involves:
  - ❖ designing a simulation for a given situation
    - • identifying tools to be used
    - • defining a trial and the number of trials
    - • determining data recording methods
  - ❖ carrying out the simulation and recording outcomes
  - ❖ selecting and using appropriate displays and measures
  - ❖ communicating findings in a conclusion.

Simulation Tools

Simulations

Situations with outcomes that depend upon probability may be examined by studying repeated results (relative frequency) but often the situation cannot or should not be repeated enough times to get an accurate picture of the probability of each outcome. It could be because the situation is expensive to repeat or because it involves the destruction of some component of the situation. For example the likelihood of an aircraft crashing when an engine fails.

In these cases we look to carry out a realistic simulation so that the probability of each outcome is the same as in the original situation and then we use the repeated results of our simulation to draw appropriate conclusions or results about our situation.

The simulation needs to be designed so that it models as accurately as possible the actual situation. All outcomes will need to be identified and an estimate made as to the probability of each possible outcome.

To generate the results we use a tool that has the same probability of each outcome in a nondestructive and relatively fast setting. Examples include

- a deck of cards
- dice (4, 6, 8 or 10 sided)
- a spinner with the appropriate divisions
- the flip of a coin
- random number tables
- a random number generator on a calculator
- a computer programme such as Numbers or Excel with a random number generator.

All of these tools enable the probabilities to be generated a large number of times and therefore they can simulate the situation.

Physical Probability Tools

We can use physical devices such as cards, dice, the flip of a coin or a spinner when the probability of the desired outcome can be matched to the theoretical probability of the device.

For example, if we were attempting to simulate something with a probability of $\frac{1}{3}$, we could simulate with cards (take out one suit), a die (1 and 2 represents one outcome, 3 and 4 the second and 5 and 6 the third outcome), or a spinner (each sector is 120°) but it would not be easy to use coins to simulate $\frac{1}{3}$.

Extra Notes

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Achievement – Use random number tables or your calculator to generate the following random numbers. Calculate the mean of your answers.

5. 20 random numbers from 1 to 8.
Calculate the mean of your numbers.

6. 30 random numbers from 1 to 13.
Calculate the mean of your numbers.

7. 30 random numbers from 01 to 99.
Calculate the mean of your numbers.

8. 30 random numbers from 10 to 16.
Calculate the mean of your numbers.

Example

The polygraph test is a lie detector test that is not used in New Zealand. Manufacturers claim it is between 70% and 90% accurate. Assuming the lower figure this means that 70% of the time a guilty suspect is shown as lying by the polygraph they are guilty. However 30% of guilty suspects will not be identified as lying. Also 70% of the time an innocent suspect is telling the truth the polygraph will agree but 30% of the time an innocent suspect will wrongly be identified as lying and therefore presumed guilty.

The police investigating a crime know that one of four suspects are guilty. If they were able to give a lie detector test, find the probability that when the polygraph says a suspect is guilty, that they really are guilty.

"Does this mean the suspect is lying?"
"No, he is just shaking so much he has triggered the building's Seismograph."

We need to simulate which of the suspects are guilty or innocent and then which the polygraph concludes are guilty or innocent.

The required probability is the fraction of guilty suspects found guilty out of all the suspects found guilty by the polygraph.
For each trial we will determine if a suspect is guilty (Prob. = 0.25) by generating a random number 1 to 4 where 1 to 3 are innocent and 4 is guilty. Then we will check if the test is accurate. A random number 1 to 10 will then determine if the test says they are lying.

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Achievement – Simulation 1, Mini Lotto

Description

Lotto has 40 balls and to win you must correctly select six in any order. In this Mini Lotto there are only six numbers. You have to design and run your own simulation, so that you can estimate the probability of winning when selecting two numbers out of the six.

Instructions

You have to decide whether you will use a calculator, random number or a die. Design and carry out your own simulation recording the results below. The instructions should be clear enough so that someone else can carry out the simulation. Use the same pair of winning numbers for all the simulations. Remember no number can appear more than once, so a result of 5, 5 (for example) is impossible.

Your Design

Your pre-selected winning two numbers are ______ and ______.

Trial	1	2	3	4	5	6	Win Y/N	Trial	1	2	3	4	5	6	Win Y/N
1								26							
2								27							
3								28							
4								29							
5								30							
6								31							
7								32							
8								33							
9								34							
10								35							
11								36							
12								37							
13								38							
14								39							
15								40							
16								41							
17								42							
18								43							
19								44							
20								45							
21								46							
22								47							
23								48							
24								49							
25								50							

Probability of winning with Mini Lotto.

Theoretical probability

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Merit – Simulation 3, Doner Kebabs

Description

Eric loves Doner Kebabs but he also likes variety. He does not choose the two sauces that go into each kebab but selects them purely at random. There are four mild sauces (tomato sauce, barbecue sauce, hummus and yoghurt sauce) and two hot spicy sauces (hot chilli and mild chilli with garlic sauce). No single sauce is selected twice.

Task

Estimate by simulation the probability that

- Eric's kebab has two mild sauces on it.
- Eric's kebab has a mild and spicy sauce on it.

Instructions

To simulate the type of the sauce (mild or spicy) which is selected we generate a random number from 1 to 6. We could do this with a die or a calculator. If the result is a 5 or 6 then the sauce is spicy.

In selecting the second sauce there is now only five sauces still available. We need to consider if the first sauce was mild then there are three mild sauces and two spicy sauces left. If the first sauce was spicy, there is still four mild sauces and one spicy sauce left. Therefore we now generate a random number 1 to 5. We can still use a die by disregarding the result if a six is rolled or generate numbers 1 to 5 on the calculator. Depending upon the first sauce type the range of results for the second sauce will change. If the first sauce is mild, then a result of 4 or 5 out of 5 will give a spicy sauce otherwise the second sauce is also mild. If the first sauce is spicy, then a result of 5 for the second sauce will give a second spicy sauce otherwise the result is a mild sauce.

Example:

Trial number	First result 1 – 6.	First sauce	Second result 1 – 5.	Second sauce	Two sauces
Ex.	3	M	5	S	MS

Trial Data 1 – 40

Trial number	First result 1 – 6.	First sauce	Second result 1 – 5.	Second sauce	Two sauces
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
14					
15					
16					
17					
18					
19					
20					

Trial number	First result 1 – 6.	First sauce	Second result 1 – 5.	Second sauce	Two sauces
21					
22					
23					
24					
25					
26					
27					
28					
29					
30					
31					
32					
33					
34					
35					
36					
37					
38					
39					
40					

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Computer Spreadsheets

Using Spreadsheets

We can use a spreadsheet programme such as Microsoft Excel, Open Office or Numbers (Mac) to generate a large number of results and even to add up how many of each result or calculate means etc. This requires putting formulae into the spreadsheet programme but as this is an internal Achievement Standard it is possible to adapt an existing spreadsheet or to write your own.

The instructions here are for Microsoft Excel but they can be adapted to any spreadsheet programme. If you do not have Excel on your computer then the authors suggest you look at the free software Open Office.

A spreadsheet is a large grid where numbers or formulae are placed in each cell of the grid. At the top of each column is a letter and on the left of each row is a number. If you wanted to refer to the cell in column 2 and row three you would call this Cell B3.

If, in any of these three spreadsheet programmes you wanted to put an equation to generate a random number between 1 and 8 inclusive

i.e. $R_{1 \text{ to } 8}$ = Integer of (1 + 8 x random number)

then type in the cell

= INT(1 + 8*RAND())

Using an Existing Spreadsheet

Download 'Simulation Template' from the NuLake Ltd website www.nulake.co.nz under 'Downloads', 'Year 12', 'IAS 2.13'.

Open the spreadsheet in your spreadsheet programme. This is an Excel spreadsheet but it should open with Open Office and Numbers. The authors cannot guarantee that each version of each programme will work.

At the bottom of the screen should be references to each sheet of the spreadsheet. Some are called Random 2 etc. and these may be useful when you want to simulate situations where you have two (or more) possible outcomes.

Select a sheet that you can adapt to assist you in generating data for your simulation. If you need more than 100 pieces of data select A99 to B101. Release the mouse button and grab the bottom right corner and drag it down as far as required.

You will also have to amend all the formulae in column D so they look beyond B$101 to say B$201.

The cell could have a name B101 or B$101. The advantage of the $ sign is it means the reference to position 101 will not change if the formula is moved around.

Microsoft Excel
Copyright: Microsoft Corporation

Numbers
Copyright Apple Inc.

OpenOffice
Copyright Apache

	A	B
95	94	1
96	95	1
97	96	2
98	97	2
99	98	2
100	99	2
101	100	2
102		

Select the last few cells you want to repeat. Then drag down the bottom right handle as far as required.

B	C	D
ndom #	Count	150
1	Number of	
1	1	=COUNTIF(B$2:B$101, C3)
1	2	
2		

If you change the number of random numbers you will need to amend every formula on that page.

Merit/Excellence – Simulation 8, Beating the Odds

Description

In the game of Roulette there are 38 possibilities for the ball to come to rest on, 18 red, 18 black and 2 green. The probability that the outcome is red is $\frac{18}{38}$ and if you bet on red (or black) and win, you will double your money by getting your bet back and winning the same amount. If you bet \$7 you will be paid your original bet plus \$7 winnings.

Vanessa has decided she can make sure she will win by betting just on the red. She bets \$1 and if she wins she pockets her winnings of \$1 and starts again. If she loses her first bet, she doubles her bet to \$2 and if she wins on the second time she uses the winnings of \$2 to pay herself back the \$1 she lost on the first round and pockets the remaining \$1 and starts again at \$1.

If she loses on the second round she again doubles her bet now up to \$4. If she wins she uses the \$4 winnings to pay back the \$3 she has lost in the first two rounds and pockets the remaining \$1 and starts again at \$1.

She continues doubling her bet every times she loses so that she covers all her losses and still is able to make a profit of \$1. The Casino has a limit of \$50 on a single bet so in the unlikely event she loses six times in a row she will lose the \$1, \$2, \$4, \$8, \$16 and \$32 a total of \$63 and have to start again at \$1.

Vanessa figured it was highly unlikely that she would lose six times in a row for this loss to apply.

Assumption

We are assuming that the game is not biased and that every outcome is equally likely.

Task

Design a simulation to see what Vanessa is likely to win (or lose) using her method. If you are using a computer, repeat your simulation of 100 trials at least ten times to see the pattern of the results. If you are using random number tables or a calculator then share your results with nine other students to get a pattern of results.

Instructions

To undertake this simulation generate a random number between 1 and 38 to represent the 38 outcomes. Red represents 18 outcomes, so if the number is under 19 then Vanessa wins. In reality the first 18 numbers are not red but we are just assigning 18 out of 38 outcomes to represent winning outcomes.

Outcome = Integer value of (1 + 38 x Random number)

On a computer use

Outcome = INT(1 + 38*RAND())

If the result is under 19 we win \$1 so we copy the formula

= If(INT(1 + 38*RAND()) < 19, 1, 0)

down column B of our spreadsheet.

We want to record a \$1 if any of the first six spins is a win and $^-$\$63 (equal the losses \$1 + \$2 + ... + \$32) if after six spins we have not won and must start again.

Therefore on the second to sixth spin we need to check if we have won previously, or if we've won on this spin. If previous roll was a loss then we roll again and check if we now win. If we have already won we do nothing more. This becomes the logical statement;

= If the previous roll was a loss, roll again and record 1 for a win or 0, if we have already won leave blank.

= IF(B2 = 0, (IF(INT(1 + 38*RAND()) < 19, 1, 0)), "")

There is an example of a template on the 'Simulation Template' from NuLake Ltd, website www.nulake.co.nz under 'Downloads'.

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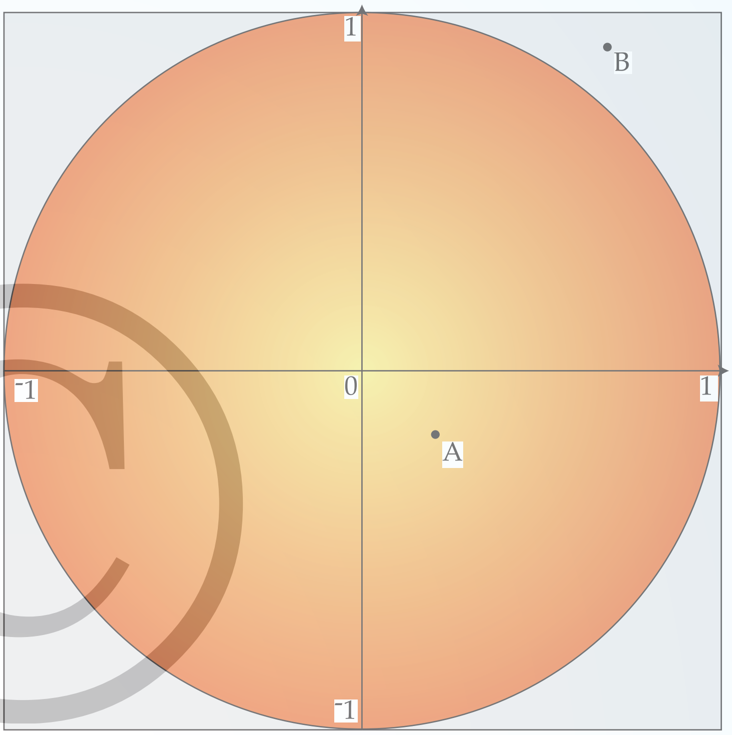
Excellence – Simulation 10, Squaring the Circle – an estimation of Pi (π)

Description



The square on the right has an area of

$$\text{Area (square)} = 2 \times 2 = 4 \text{ units}^2.$$

The circle has an area of

$$\text{Area (circle)} = \pi r^2 = \pi$$

Therefore the ratio of the area of the circle to the area of the square is

$$\text{Ratio} = \frac{\pi}{4}$$

If we randomly fire shots at the rectangle the ratio of the number of shots in the circle to the number of shots in the entire square (includes shots in the circle) will be $\frac{\pi}{4}$.

We can check visually whether a shot is in the circle or just in the square. Mathematically we can also check by calculating the distance from the centre.

If we look at A (0.2154, $^-$0.2005) we can calculate its distance from (0, 0) using Pythagoras.

$$\text{Distance A} = \sqrt{(0.2154)^2 + (^-0.2005)^2} = 0.2943$$

As this is less than the radius of 1 it is within the circle.

Position B is (0.7162, 0.9112), we can calculate its distance from (0, 0) using Pythagoras.

$$\text{Distance B} = \sqrt{(0.7162)^2 + (0.9112)^2} = 1.1590$$

which is outside the radius of 1.

Task

We want to estimate the value of Pi (π) by randomly firing shots in the 2 x 2 square and looking at the probability that the shot lands in the circle.

Instructions

You need to describe how you would answer this probability problem using simulation. The simulation must be described in sufficient detail to enable someone else to replicate your results. Remember to calculate the distance of each shot from the centre.

As part of the description you should explain

- What constitutes a trial.
- How you determine the number of trials required.
- How the results are recorded and exactly what is recorded.

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Answers

Page 6

1. Roll a die and if it is a six ignore it and roll again. If the result is a one = 'Gift'.
   If it is a two to five = 'No gift'.
   Roll a second time and again record if you have a one = 'Gift' or a two to five = 'No gift'.

   Repeat a large number of times. The required probability is the number of times you have got a TWO roll result of 'No gift' divided by the number of rolls.
2. Use a spinner with twenty divisions. Each division is 18°. One of the divisions represents going by car, five of them represent the bus and 4 of them represent walking. The remaining ten divisions represent biking.
3. As there are six possible outcomes the best physical model would be a six-sided die. Roll the die the first time and note the result. Then roll for the second day and if it is a repeated result it is ignored and the die rolled again. For each day if the result is a five or six it represents buying lunch otherwise a lunch from home.
4. Each intersection represents two outcomes so either the toss of a coin or cutting a pack of cards and look at the colour only. This is done twice to see the result. Only one combination represents home.

Page 9

5. Usually 2 or 3 of each number. Mean is likely to be between 3.5 and 5.5 (Theory = 4.5).
6. Usually 2 of each number. Mean is likely to be between 5.5 and 8.5 (Theory = 7).
7. Usually 0 or 1 of each number. Mean is likely to be between 39 and 61 (Theory = 50).
8. Usually 3 or 5 of each number including 10 and 16. Mean is likely to be between 12 and 15 (Theory = 13).

Page 11

Simulation 1

Mini Lotto

In the description the following points will be needed:

- Two numbers need to be selected first.
- Randomly generate two numbers from 1 to 6 on a calculator Intg(1 + 6 x Ran#) and subsequently pressing the EXE key will generate random numbers from 1 to 6.
- Discard any repeated numbers in the two random numbers.
- Record a ✔ if the two numbers are correct and a ✖ if incorrect.
- Repeat this a large number of times (e.g. 50).

The experimental probability should be between 0 and 0.12.

The theoretical result is given by the number of different ways two numbers can be selected divided by 2 as the order of selection is not important. There are 6 selections possible for the first number leaving 5 for the second number.

$$\text{Ways} = \frac{\text{Select 1st} \times \text{Select 2nd}}{2}$$

$$= \frac{6 \times 5}{2}$$

$$= 15$$

$$P(\text{win}) = \frac{1}{15} \ (= 0.067 \text{ to 3 dp})$$

Page 12

Simulation 2

Lego toys

Expected number of visits is between 14 and 15 but results will vary around these figures.

The largest number of visits is 32 and this is a lot more than expected.

Author's results were 15.8 visits.

#	Toy 1	Toy 2	Toy 3	Toy 4	Toy 5	Toy 6	CF
1	✓	✓	✓	✓	✓	✓	17
2	✓	✓	✓	✓	✓	✓	21
3	✓	✓	✓	✓	✓	✓	10
4	✓	✓	✓	✓	✓	✓	16
5	✓	✓	✓	✓	✓	✓	14
6	✓	✓	✓	✓	✓	✓	15
7	✓	✓	✓	✓	✓	✓	12
8	✓	✓	✓	✓	✓	✓	10
9	✓	✓	✓	✓	✓	✓	17
10	✓	✓	✓	✓	✓	✓	11
11	✓	✓	✓	✓	✓	✓	19
12	✓	✓	✓	✓	✓	✓	13
13	✓	✓	✓	✓	✓	✓	12
14	✓	✓	✓	✓	✓	✓	9
15	✓	✓	✓	✓	✓	✓	15
16	✓	✓	✓	✓	✓	✓	17
17	✓	✓	✓	✓	✓	✓	27
18	✓	✓	✓	✓	✓	✓	7
19	✓	✓	✓	✓	✓	✓	32
20	✓	✓	✓	✓	✓	✓	17
21	✓	✓	✓	✓	✓	✓	14
22	✓	✓	✓	✓	✓	✓	17
23	✓	✓	✓	✓	✓	✓	19
24	✓	✓	✓	✓	✓	✓	18
25	✓	✓	✓	✓	✓	✓	17